ICTP Diploma Programme
Course on Advanced Quantum Mechanics

Problem Set 1: Academic Year 2011-2012

Write out solution clearly and concisely. Diagrams/graphs welcome. Number pages and problems clearly.

Problem 1: Linear operators

Which of the following operators acting on functions is a *linear operator*? (By $\psi'(x)$ we mean the derivative with respect to x.)

1. $O_1\psi(x) = x^3\psi(x)$
2. $O_2\psi(x) = x\psi'(x)$
3. $O_3\psi(x) = \lambda\psi^*(x)$
4. $O_4\psi(x) = e^{\psi(x)}$
5. $O_5\psi(x) = \psi'(x) + a$

Problem 2: More wave-packets

Consider the following wave-packets for a non-relativistic free-particle in one-dimension:

$$\psi(x,t) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}}\, \phi(k) e^{i(kx - E_k t/\hbar)} \tag{1}$$

where $E_k = \hbar^2 k^2/2m$, and the weight functions are either i a Lorentzian $\phi(k) = C/(\Gamma^2 + (k-k_0)^2)$ or ii a decaying exponential $\phi(k) = De^{-|k-k_0|/\Gamma}$ (here C and D are appropriate normalization constants, while Γ controls the width in k-space of the function $\phi(k)$). Calculate, for both choices, the wave-function at time $t = 0$, i.e., the Fourier transform of the corresponding $\phi(k)$ and observe the very different behaviour for large x. Calculate, for both choices, the uncertainties Δx and Δp. Notice how difficult would be the calculation of the wave-packets for $t > 0$ (the k-integrals become very complicated), contrary to the Gaussian case we did in class.

Problem 3: Current density for one-dimensional wavefunctions

Calculate the probability-density current $J(x)$ for: a) a wavefunction of the form $\psi(x) = Ae^{ikx} + Be^{-ikx}$, b) a wavefunction of the form $\psi(x) = u(x)e^{ikx}$ with $u(x)$ a *real* function. c) a real wavefunction of the form $\psi(x) = Ae^{-k|x|}$, with $k > 0$.

Problem 4: Uncertainty in position and momentum for a particle in a infinite box

Calculate the uncertainty in position

$$\Delta x = \sqrt{\langle\psi_n|(x - \langle\psi_n|x|\psi_n\rangle)^2|\psi_n\rangle} \ ,$$

for the eigenstates $\psi_n(x)$ of a one-dimensional particle in a box (infinite potential outside the box). Calculated also the uncertainty in momentum Δp, using the fact that $\langle p\rangle = 0$ and $p^2 = 2mH$. Calculate the product $\Delta x \Delta p$ for general n.

Problem 5: Particle in a box with twisted boundary conditions

Solve the problem of a particle in a one-dimensional box in $[0, L]$ assuming now, as boundary conditions, that $\psi(L) = e^{i\phi}\psi(0)$, with $\phi \in [0, 2\pi)$ an arbitrary real phase, rather than $\psi(0) = \psi(L) = 0$ as for the case done in class (corresponding to an infinite potential outside the box). Observe what happens to the spectrum and the degeneracies of each level by varying $\phi \in [0, 2\pi]$.

Problem 6: Particle in a finite square well

Solve the Schrödinger problem for a particle in a one dimensional potential which is equal to $-V_0 < 0$ in a region $[-a, a]$ and zero outside of it. Do both the bound state case, $-V_0 < E < 0$ and the scattering state case, $E \geq 0$. Study in particular the number of bound state as a function of the depth V_0 of the well. Study also the transmission amplitude as a function of E for a particle approaching the well from the left with a wavefunction $e^{ikx} + r_k e^{-ikx}$.

Problem 7: Transmission through a square barrier

Solve the Schrödinger problem for a particle in a one dimensional potential which is equal to $+V_0 > 0$ in a region $[-a, a]$ and zero outside of it. Concentrate in particular on the region $0 < E < V_0$ and calculate the transmission amplitude as a function of E for a particle approaching the barrier from the left with a wavefunction $e^{ikx} + r_k e^{-ikx}$.

Problem 8: Time-evolution for a particle in a infinite box

Suppose a particle is in an infinite well located in $[-a, +a]$ and is initially prepared in the state

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}(\phi_1(x) + \phi_2(x)) \ ,$$

where $\phi_{1,2}(x)$ are the lowest two eigenstates for the particle in the box. Write down the solution $\psi(x, t)$ of the Schrödinger equation at all later times t, and evaluate on it the expectations values: a) $\langle\psi(t)|x|\psi(t)\rangle$, b) $\langle\psi(t)|p|\psi(t)\rangle$, c) $\langle\psi(t)|x^2|\psi(t)\rangle$, d) $\langle\psi(t)|p^2|\psi(t)\rangle = 2m\langle\psi(t)|H|\psi(t)\rangle$. How are the average position and momentum depending on t? How is the average energy depending on t? How are the uncertainties in x and p depending on t?