Advanced Electromagnetism lecture notes

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Suggested textbooks: J.D. Jackson "Classical Electrodynamics" (The third edition is adopted for the references given in the abstract of each lecture)

How to use this material

The material presented in this first version of the notes of the AE course should be considered as complementary to the registered lectures and are not intended to substitute the reference book . For each lecture, first it is given the abstract of the lecture (the one you find in the Moodle's web page of the AE course) then follows a concise (and not exhaustive) report of the lecture content. Symbols and equations are written as in the registered lecture or in the book. Readers should pay attention to the units (SI or Gaussian) used to write equations. Finally I thank Mukiibi Daniel and Solomon Admasu Alemayehu for providing me with their handwritten notes (and pictures sometimes shown in the notes) that have been really helpful for selecting the contents that are presented in this AE lecture notes.

L.1 Lorentz's force

the Lorentz's force is introduced and an example is given of a charge particle moving in a region with uniform and constant electric and magnetic fields.

The motion of a particle of mass m and charge q in a region crossed by an electric $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is described by the Lorentz force (SI units):

$$\mathbf{F}(\mathbf{x},t) = q[\mathbf{E}(\mathbf{x},t) + \mathbf{v} \times \mathbf{B}(\mathbf{x},t)]. \tag{1}$$

Please note that with Gaussian units the previous equation takes the form:

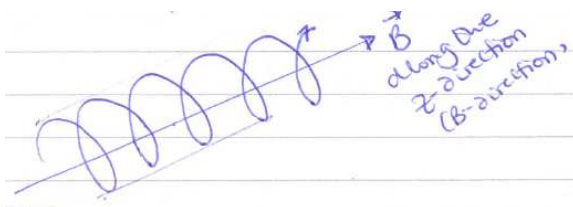
$$\mathbf{F}(\mathbf{x},t) = q[\mathbf{E}(\mathbf{x},t) + \frac{\mathbf{v}}{c} \times \mathbf{B}(\mathbf{x},t)]. \tag{2}$$

Let's now consider uniform constant electric and magnetic fields. For a zero electric field $E = 0$, the trajectory of the particle consists in a series of spirals along the direction of the $\mathbf{B}$ field (that we can choose as z direction)

$$x(t) = x_0 + R\sin(\omega_B t + \alpha) \tag{3}$$
$$y(t) = y_0 + R\cos(\omega_B t + \alpha) \tag{4}$$
$$z(t) = z_0 + v_{0z} t \tag{5}$$

where v_{0z} is the initial velocity along the z-direction and $\omega_B = \frac{qB}{m}$. For a

Figure L.1: *trajectory of a charged particle in a region of non-zero magnetic field.*



non-zero electric field $E \neq 0$ the equation of motion reads

$$m\ddot{x} = \frac{qB}{c}\dot{y} \tag{6}$$
$$m\ddot{y} = -\frac{qB}{c}\dot{x} + qE_y \tag{7}$$
$$m\ddot{z} = qE_z \tag{8}$$

by integrating the equation of motion we find:

$$
\begin{aligned}
z(t) &= \frac{qE_z}{2m}t^2 + v_{0z}t + z_0 && (9)\\
y(t) &= \frac{A}{\omega}(\cos\omega t - 1) && (10)\\
x(t) &= \frac{A}{\omega}(\sin(\omega t + \alpha) + \frac{cE_y}{v_0}\omega t. && (11)
\end{aligned}
$$

The previous equations show that to accelerate a charge we need to apply an electric field but to make it rotating we need to apply a magnetic field: **B** field only changes the direction of the particle and do not contribute to the linear acceleration that is provided by **E** field.

L.2 Maxwell's equations

In the first part of this lesson, Maxwell's equations are reminded to the students first by explaining their meaning and then by expressing them through mathematical symbols (integral forms are used). Meanings of flux and circuitation of a field are introduced through an analogy with water currents and whirlpools. The minus sign appearing in the law ("lenz law") $\int \mathbf{E}\cdot \mathrm{d}\ell = -\mathrm{d}/\mathrm{dt}(\int \mathbf{B}\mathrm{d}\mathbf{S})$ *is associated to the energy conservation. It is then remarked that Gauss law is completely equivalent to Coulomb law for the force between two charges. Next, Gauss and Stokes theorems are reminded in order to write Maxwell's equations in differential form. While E field is generated by charges no monopole exists for the magnetic field i.e.* $\nabla\cdot\mathbf{B} = 0$. *The latter is generated by currents* **J** *like Oersted first noticed. The second part begins with the continuity equation for the electric charge that leads to the concept of local conservation. Then follows a discussion on the units used in this course. The first set is the standard SI system, the second set is the given by the Gauss units (CGS). It is then mentioned the Ampere's law for the force between two wires.*

The Maxwell equations in integral form are:

$$\text{[Gauss law]} \int_S \mathbf{E}\cdot\mathbf{n}\, dS = \frac{1}{\epsilon_0}\int_{V_S} \rho(\mathbf{r})\, dV_S \tag{12}$$

$$\oint_C \mathbf{E}\cdot d\ell = -\frac{d}{dt}\int_{S_C} \mathbf{B}\cdot\mathbf{n}\, dS \tag{13}$$

$$\int_S \mathbf{B}\cdot\mathbf{n}\, dS = 0 \tag{14}$$

$$c^2\oint_C \mathbf{B}\cdot d\ell = \frac{d}{dt}\int_{S_C} \mathbf{E}\cdot\mathbf{n}\, dS + \frac{1}{\epsilon_0}\int_{S_C} \mathbf{J}\cdot\mathbf{n}\, dS_C \tag{15}$$

Thanks to Gauss and Stoke's theorems we can derive, from the previous equations, the differential form of Maxwell equations (SI units):

$$\nabla\cdot\mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0} \tag{16}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \tag{17}$$

$$\nabla\cdot\mathbf{B} = 0 \tag{18}$$

$$c^2(\nabla\times\mathbf{B}) = \frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\epsilon_0}\mathbf{J}. \tag{19}$$

Maxwell equations together with Lorentz force are jointly referred to as the laws of electromagnetism.

Exercise 1

Prove the Continuity equation using Maxwell equations.

Let's take the time derivative of Gauss law

$$\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = \frac{\partial}{\partial t}(\frac{\rho}{\epsilon_0}),$$

and the divergence of the last of Eqs. 21-24:

$$c^2\nabla\cdot(\nabla\times\mathbf{B}) = \nabla\cdot(\frac{\partial}{\partial t}\mathbf{E}) + \frac{\nabla\cdot\mathbf{J}}{\epsilon_0}$$

Now, the left hand side of the previous equation is always zero:

$$0 = \nabla\cdot(\frac{\partial}{\partial t}\mathbf{E}) + \frac{\nabla\cdot\mathbf{J}}{\epsilon_0} = \frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) + \frac{\nabla\cdot\mathbf{J}}{\epsilon_0}.$$

Thus we can derive the Continuity equation:

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \tag{20}$$

that expresses the conservation of charge.

Remark: units

In the Gauss system of units Maxwell equations are written as:

$$\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{r}) \tag{21}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{22}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{23}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}. \tag{24}$$

The Coulomb's law force:

$$\mathbf{F} = k\frac{q_1 q_2}{r^3}\mathbf{r} \tag{25}$$

In Gauss system $k = 1$ while in SI units $k = 1/4\pi\epsilon_0$ where $\epsilon_0 = \frac{1}{4\pi c^2} \times 10^7 \frac{As}{Vm}$.

Ampere's law force:

$$d\mathbf{F} = kI_1 I_2 \frac{d\mathbf{s}_1 \times (d\mathbf{s}_2 \times \mathbf{r}_{12})}{r_{12}^3} \tag{26}$$

In Gauss units $k = 1/c^2$. In SI units $k = \frac{\mu_0}{4\pi}$.

Lorentz force

$$\mathbf{F} = k\frac{q}{c}\mathbf{v} \times \mathbf{B} \tag{27}$$

In Gauss units $k = 1$. In SI units $k = c$.

L.3 Maxwell's equations in macroscopic media

This lecture is dedicated to Maxwell's equations in macroscopic media. Currents and charges are splitted into two terms ("external" and "induced") e.g: $\mathbf{J} = \mathbf{J_{ext}} + \mathbf{J_{ind}}$. *This lead to a rewriting of Maxwell's equation in terms of a new set of fields:* $\mathbf{D}, \mathbf{H}, \mathbf{B}, \mathbf{E}$ *that keep into account the electric and magnetic answers of materials to applied fields [Jackson, Introduction I.4]. Concept of polarization and magnetization are introduced. It is seen that macroscopic fields such as D can be expanded in series containing dipole, quadrupole....terms [Jackson, Eq. I.9]. Next, boundary conditions at interfaces between different media are obtained by applying Maxwell's equations for the set* $\mathbf{D}, \mathbf{H}, \mathbf{B}, \mathbf{E}$ *[Jack. Eqs. I.17-18 and I.19-20]. Then electrostatics and magnetostatics law are considered for introducing auxiliary fields called scalar and vector potentials that will be later very helpful for a Lagrangian*

description of electromagnetism. Finally, concerning the "reality" of the vector potential, the Aharonov-Bohm effect is briefly mentioned.

The effects of applying **E** and **B** fields to some medium causes the appearance of induced charges and currents:

$$\mathbf{J} = \mathbf{J}_{ext} + \mathbf{J}_{ind} \tag{28}$$

$$\rho = \rho_{ext} + \rho_{ind}. \tag{29}$$

The induced current $\mathbf{J}_{ind}$ is then responsible for polarization and magnetization phenomena that are described through the polarization **P** and magnetization **M** fields: $\mathbf{J}_{ind} = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}$. The induced charge is also related to the polarization by $\rho_{ind} = -\nabla \cdot \mathbf{P}$. By substituting the previous definitions into the Maxwell equations we find (the other two equation are unchanged):

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{ext} \tag{30}$$

$$\nabla \times (\frac{\mathbf{B}}{\mu_0} - \mathbf{M}) = \mathbf{J}_{ext} + \frac{\partial(\epsilon_0 \mathbf{E} + \mathbf{P})}{\partial t} \tag{31}$$

where ϵ_0 and μ_0 are related to each other by $\frac{1}{c^2} = \epsilon_0 \mu_0$. We can introduce two new fields **D** and **H**, known as electric displacement and magnetic field,[1]

and write Maxwell equations in a macroscopic medium:

$$\nabla \cdot \mathbf{D} = \rho_{ext} \tag{32}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_{ext} + \frac{\partial \mathbf{D}}{\partial t}. \tag{33}$$

A general definition of the **D** field can be obtained by including terms beyond the polarization (dipole) term:

$$D_\alpha = \epsilon_0 E_\alpha + P_\alpha - \frac{\partial Q_{\alpha\beta}}{\partial t} + ... \tag{34}$$

Moreover note that the potential of any charge distribution can be expanded in a similar way. In case of the gravitational field the first correction is the quadrupole term and no dipole is present. If P_α is small we can introduce the dielectric susceptibility $\hat{\chi}_e$ and then **D** and **E** are proportional

$$\mathbf{D} = \epsilon_0 (1 + \hat{\chi}_e) \mathbf{E}, \tag{35}$$

and similarly for the magnetic field: we can introduce a permeability $\hat{\chi}_m$ such that $\mathbf{B} = \mu \mathbf{H}$ and $\mu = \mu_0 (1 + \hat{\chi}_m)$. At the interface between two media

[1] **B** is called the magnetic induction, **E** is the electric field.

1 and 2 the fields $\mathbf{D}$, $\mathbf{H}$ may show a discontinuity. Indeed it is possible to demonstrate that for the normal component of th $\mathbf{D}$ field

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \sigma \tag{36}$$

where σ is a surface density of charge. On the contrary, the $\mathbf{B}$ field is always continuous

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = 0. \tag{37}$$

Finally for the tangential components of $\mathbf{E}$ and $\mathbf{H}$:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{k} \tag{38}$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{39}$$

Let's now consider the set of Maxwell equations in vacuum:

$$\text{Electrostatics} \qquad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = 0 \tag{40}$$

$$\text{Magnetostatics} \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{\epsilon_0 c^2}\mathbf{J} \tag{41}$$

From the previous equations it is possible to see that the fields $\mathbf{E}$, $\mathbf{B}$ can be expressed in terms of a new set of fields named "auxiliary fields". Indeed the fact that $\mathbf{E}$ is irrotational and $\mathbf{B}$ is solenoidal means that we can introduce a scalar potential Φ and a vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\nabla\Phi \tag{42}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{43}$$

Exercise

Verify that:
(i) $\nabla \times \nabla\phi = 0$
(ii) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

L.4 Gauss's law

Contents of this lecture: - Integral and differential forms of Gauss's law [Jackson, Sec. 1.3 and 1.4]. - Stokes's theorem for the electric field. - Brief introduction to the Ohm's law. - Applications of the Gauss's law: Electric field generated by a wire (linear charge), by a sheet of charge and by a plane capacitor. - Potential generated by N point-like charges. - Poisson's equation [Jackson, Sec. 1.7]. - Energy density of the electric field [Jackson, Sec. 1.11]. - Dipole field.

Coulomb law is equivalent to Gauss law.

If we take a point charge q and a spheric surface S centred in q, then by assuming Coulomb law we get

$$\int \mathbf{E} \cdot \mathbf{n} \, dS = \int \frac{q}{4\pi\epsilon_0 r^2} \frac{\mathbf{r}}{r} \cdot \mathbf{n} \, dS$$

that becomes

$$\int \frac{q}{4\pi\epsilon_0 r^2} r^2 \, d\Omega = \frac{q}{4\pi\epsilon_0} \int d\Omega = \frac{q}{\epsilon_0}.$$

The viceversa can also be easily demonstrated (Exercise).

Work of the electric field (electrostatic)

Let's consider $\nabla \times \mathbf{E} = 0$. By Stokes theorem this means $\oint_C \mathbf{E} \cdot d\ell = 0$. Now the work to move the charge q from a point A to a point B is given by

$$W = -\int_A^B \mathbf{F} \cdot d\ell = -q \int_A^B \mathbf{E} \cdot d\ell = q \int_A^B \nabla\Phi \cdot d\ell = q \int_A^B d\Phi = q(\Phi_B - \Phi_A).$$

Ohm's law

The microscopic form of Ohm's law is:

$$\mathbf{J} = \hat{\sigma}\mathbf{E} \tag{44}$$

where $\mathbf{J}$ is the current density, $\hat{\sigma}$ is the electric conductivity. In an isotropic material $\sigma = 1/\varrho$ where ϱ is the resistivity. Let's consider a wire and let's calculate the potential drop between two points 1 and 2 along the wire:

$$V_{12} = \phi_1 - \phi_2 = -\int_1^2 d\phi = -\int_1^2 \nabla\phi \cdot d\mathbf{s} \tag{45}$$

$$= \int_1^2 \mathbf{E} \cdot d\mathbf{s} = \int_1^2 \frac{\mathbf{J}}{\sigma} \cdot d\mathbf{s} \tag{46}$$

Now, since $JA = I$ and $R = \varrho\frac{\ell}{A}$ where R is the resistance and A the area of wire section, it follows:

$$V_{12} = IR. \tag{47}$$

Electric field generated by a wire

Consider an infinitely long straight wire with a charge density per unit length λ. The electric field, by simmetry reasons will have a radial simmetry in the plane orthogonal to the wire. By Gauss theorem:

$$E \cdot 2\pi r = \frac{\lambda}{\epsilon_0}$$

and hence

$$E = \frac{\lambda}{2\pi\epsilon_0 r}. \tag{48}$$

Electric field generated by a sheet of charge

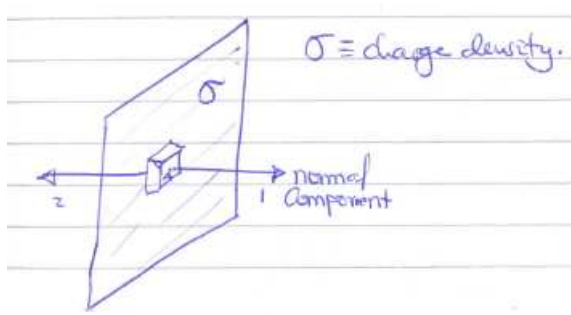
Consider a sheet of charge having a charge density σ. By symmetry reasons, on each side of the plane, the electric field will be directed along the normal direction to the plane. By Gauss theorem it follows

$$EA + EA = \frac{\sigma A}{\epsilon_0}$$

and hence

$$E = \frac{\sigma}{2\epsilon_0}. \tag{49}$$

Figure L.2: *Electric field generated by a sheet of charge having a charge density σ*



Electrostatic potential energy

Let's consider N point-like charges q_i. Then each charge will have a potential energy $W_i = q_i \Phi_i(\mathbf{r})$, where

$$\Phi_i(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1, j\neq i}^{N} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{50}$$

and

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \sum_{j<i} \frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{51}$$

$$= \frac{1}{8\pi\epsilon_0} \int d^3V \int d^3V' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{52}$$

$$= \frac{1}{2} \int \rho(\mathbf{r})\phi(\mathbf{r})\ d^3V. \tag{53}$$

Now using Poisson equation $\nabla^2\Phi = -\rho/\epsilon_0$ we can write

$$W = -\frac{\epsilon_0}{2} \int \Phi \nabla^2 \Phi\ d^3V.$$

By integrating per parts we then obtain

$$W = \frac{\epsilon_0}{2} \int |\nabla\Phi|^2\ d^3V = \frac{\epsilon_0}{2} \int |\mathbf{E}|^2\ d^3V, \tag{54}$$

where $w = \frac{\epsilon_0}{2}|\mathbf{E}|^2$ is the energy density or energy in a unit volume.

Dipole field

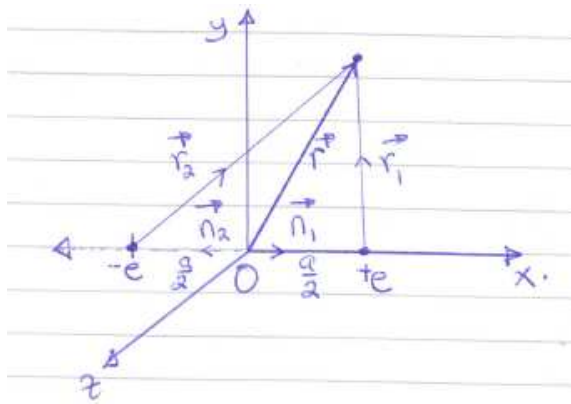
Let's consider a dipole as shown in the following figure Fig. **??** We assume the potential for an isolated charge to be $V = kq/r$. For our dipole system clearly $V = ke(1/r_1 - 1/r_2)$ where $\mathbf{r}_1 = \mathbf{r} - \frac{a}{2}\hat{\mathbf{x}}$ and $\mathbf{r}_2 = \mathbf{r} + \frac{a}{2}\hat{\mathbf{x}}$ where $\hat{\mathbf{x}}$ is the unit vector in the x-direction. Now

$$V = ke\left(\frac{1}{(r^2 + (1/2)^2 - a(\mathbf{r}\cdot\hat{\mathbf{x}}))^{1/2}} - \frac{1}{(r^2 + (1/2)^2 + a(\mathbf{r}\cdot\hat{\mathbf{x}}))^{1/2}}\right)$$

Since $a << r$ we can neglect terms in a^2 and we can use the following approximation:

$$[1 - \frac{a}{r^2}(\mathbf{r}\cdot\hat{\mathbf{x}})]^{1/2} \approx 1 + \frac{a}{2r^2}(\mathbf{r}\cdot\hat{\mathbf{x}}.)$$

Figure L.3: *A pair of charges forming a dipole* $\mathbf{p}$.



The potential generated by a dipole can then be written as

$$V(\mathbf{r}) = k\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \tag{55}$$

where we defined the dipole moment $\mathbf{p} = ea\hat{\mathbf{x}}$ of the pair of charges, and consequently $E = -\nabla V$ where

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\hat{\phi}$$

is written as

$$\mathbf{E} = \frac{2kp\cos\theta}{r^3}\hat{r} + \frac{kp\sin\theta}{r^3}\hat{\theta}. \tag{56}$$

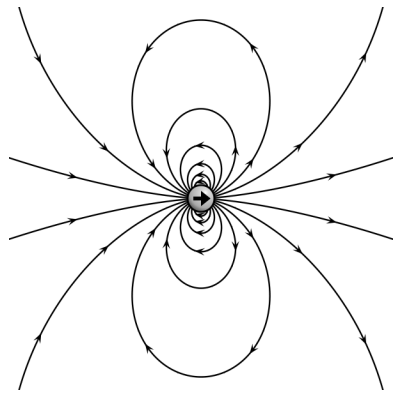
or

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3}\right) \tag{57}$$

L.5 Poisson equation

The first part of this lesson presents the solution of the Poisson equation through a Green's function approach. Dirichlet and Neumann boundary conditions are discussed (Jackson sect. 1.8, 1.9 and 1.10). Next, the method of the images is applied to find out the potential generated by a charge at a distance a from a plane put at ground (zero potential). As further example,

Figure L.4: *The dipole field.*



it is also shown how to find the potential generated by a charge with the constraint to be zero on a spherical surface at a distance x from the charge. Finally a few mathematical properties of analytic functions are discussed for the solution of two-dimensional electrostatic problems.

From the laws of electrostatics $\mathbf{E} = -\nabla\phi$ and $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ we can immediately see that $-\nabla \cdot \nabla\phi = \frac{\rho}{\epsilon_0}$, i.e

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}. \tag{58}$$

The previous equation, known as Poisson's equation, in a region where no charge distribution is present ($\rho = 0$) becomes:

$$\nabla^2\phi = 0, \tag{59}$$

that is also known as Laplace's equation.

A general solution of Eq. 58 can be found by exploiting the properties of the Green's function and Green's theorems. The Green's function associated to the Laplace's equation

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta(\mathbf{r} - \mathbf{r}') \tag{60}$$

is given in general by

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + F(\mathbf{r}, \mathbf{r}') \tag{61}$$

where F is such that $\nabla^2 F = 0$. The second Green's theorem then implies:

$$\int_V (\phi \nabla^2 G - G \nabla^2 \phi) \text{ dV} = \int_S (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}) \text{ dS}, \tag{62}$$

or equivalently:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \text{ dV} + \frac{1}{4\pi} \int_S (\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}) \text{ dS}. \tag{63}$$

Eq. 63 shows that in order to know the potential ϕ in all the space, it is required some information on the boundary. If the potential is speciefied at the surface S of the considered region then we have a so-called "Dirichlet's problem". If else the electric field is speciefied at the boundary then we have a so-called "Neumann's problem".

L.5.1 Example

As an example of Dirichlet's problem, let's consider a sphere of radius a put at ground, i.e. $\Phi_{r=a} = 0$. A charge q is placed outside the sphere at a distance x. Find the potential in every point of the space $\Phi(\mathbf{r})$.

The solution is found by the "method of the images". An image charge q' is placed inside the sphere at a distance x' from the center of the sphere. The potential is then

$$\Phi = \Phi_q + \Phi_{q'} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{|\mathbf{r} - \mathbf{x}|} + \frac{q'}{|\mathbf{r} - \mathbf{x}|} \right). \tag{64}$$

We now impose the condition

$$\Phi_{r=a} = 0, \tag{65}$$

that can also be written as

$$\Phi = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{a|1 - x/a|} + \frac{q'}{x'|1 - a/x'|} \right), \tag{66}$$

that leads to the following conditions on x' and q':

$$x' = \frac{a^2}{x} \tag{67}$$

$$q' = -q \frac{x'}{a}. \tag{68}$$

L.6 Exercises

The lesson starts with students solving a few comprehension exercises e.g writing the Maxwell equations and Lorentz force in terms of their cartesian components; showing that continuity equation follows from Maxwell equations. Moreover a proof that $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ is given. Next Gauss law is derived from Coulomb law. Then, the potential and electric field generated by a dipole moment p are calculated (spherical coordinates are used). Next exercise concerns the calculation of the capacitance for a spherical capacitor. FInally, the potential of a charge in front of a conductor is derived by means of the image charges method.

Exercise 1

$$\nabla \cdot \nabla \times \mathbf{B} = \sum_{ijk} \frac{\partial}{\partial x_i} \left(\epsilon_{ijk} \frac{\partial B_j}{\partial x_k} \right) \tag{69}$$

$$= \sum_i \left[\sum_{j<k} \left(\epsilon_{ijk} \frac{\partial B_j}{\partial x_k} \right) + \sum_{j>k} \left(\epsilon_{ijk} \frac{\partial B_j}{\partial x_k} \right) \right] = 0. \tag{70}$$

Exercise 2

From Coulomb law derive Gauss law: The electric field generated by a charge distribution $\rho(\mathbf{r})$ is

$$\mathbf{E} = k \int \rho(\mathbf{x}') \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x'. \tag{71}$$

Now we observe that

$$\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x' = -\nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \tag{72}$$

and that $\nabla \cdot \nabla = \nabla^2$. Hence

$$\nabla \cdot \mathbf{E} = -k \int \rho(\mathbf{x}') \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'. \tag{73}$$

Now $\nabla^2 \frac{1}{|\mathbf{x}-\mathbf{x}'|} = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, so that

$$\nabla \cdot \mathbf{E} = 4\pi\rho(\mathbf{x}) \tag{74}$$

$$(SI\, units) = \frac{1}{\epsilon_0} \rho(\mathbf{x}) \tag{75}$$

(Note that $k = 1$ when using Gaussian units.)

L.7 Laplace eq. in spherical coordinates

In this lesson, the Laplace equation is solved by using the separation of variables approach and cartesian coordinates for the case of a rectangular box with five sides at zero potential, while the sixth ($z = c$) has a specified potential $\Phi = V(x, y)$ [Jackson, sect. 2.9]. Next, the Laplace equation is solved in spherical coordinates which represent a more natural choice of coordinates for a large class of practical problems [Jackson, sec. 3.1]. Legendre equation and Legendre polynomials are then introduced in order to derive the general solution of the Laplace equation. Solution for the azimuthal symmetry and definition of spherical harmonics are also given [Jackson, sec 3.3 and 3.5]. In the last part of the lesson, the expansion of the Green function $1/|\mathbf{r} - \mathbf{r}'|$ in terms of spherical harmonics is used to derive a multipole expansion for the potential generated by a localized charge [Jackson, sec. 4.1]. Finally, the multipole expansion of the energy of a charge distribution is derived by expanding Φ in a Taylor series [Jackson, 4.2].

In cartesian coordinates the Laplace equation

$$\nabla^2 \Phi(x, y, z) = 0$$

could be resolved by "separation of variables" if the geometry of the problem allows to separate: $\Phi(x, y, z) = X(x)Y(y)Z(z)$. In this case we can rewrite the Laplace equation as:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

and the solution is.... ???? [to be updated]

Laplace equation in spherical coordinates

Let's find the solutions of the Laplace equation in spherical coordinates

$$\nabla^2 \Phi(r, \theta, \phi) = 0. \tag{76}$$

The previous equation can be written explicitly as:

$$\frac{1}{r}\frac{\partial^2 (r\Phi)}{\partial r^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \theta}\frac{\partial^2 (Phi)}{\partial \phi^2} \tag{77}$$

and by choosing $\Phi = \frac{\mathcal{U}(r)}{r} P(\theta)Q(\phi)$ Eq. 77 splits into the three following equations

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2 \tag{78}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P = 0 \tag{79}$$

$$\frac{d^2\mathcal{U}}{dr^2} - \frac{\ell(\ell+1)}{r^2}\mathcal{U} = 0 \tag{80}$$

The latter equation has solution:

$$\mathcal{U}(r) = Ar^{\ell+1} + Br^{-\ell}. \tag{81}$$

Using a substitution $x = \cos\theta$ Eq. 79 becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0 \tag{82}$$

which is known as the generalised Legendre equation. When $m = 0$ this equation becomes the Legendre differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + [\ell(\ell+1)]\,P = 0 \tag{83}$$

whose solutions are the Legendre polynomials $P_\ell(x)$. For $m = 0$ (azimuthal symmetry) the solution of Eq. 77 are

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^{\ell+1} + B_\ell r^{-\ell}\right]P_\ell(\cos\theta). \tag{84}$$

For $m \neq 0$ we introduce the spherical harmonics

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{\ell-m}{\ell+m}}P_\ell^m(\theta)e^{im\phi} \tag{85}$$

for example:

$$Y_{00} = \frac{1}{\sqrt{4\pi}},\ Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}...$$

and the general solution of the Laplace equation Eq. 77 is given in terms of spherical harmonics:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{m=\ell}\left[A_{\ell,m}r^\ell + B_{\ell,m}r^{-\ell-1}\right]Y_\ell^m(\theta,\phi) \tag{86}$$

Multipole expansion

Let's consider a localized charge distribution $\rho(\mathbf{r}')$. Outside the localization region the Laplace equation $\nabla^2\Phi = 0$ holds. and the potential can be written as a multipole expansion:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}q_{\ell m}\frac{Y_{\ell m}(\theta\phi)}{r^{\ell+1}}. \tag{87}$$

The problem to be solved is the determination of the constants $_{\ell m}$ in terms of the properties of the charge density $\rho(\mathbf{r}')$. Now

$$\Phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'$$

and by taking into account the expansion of the Coulomb potential in terms of spherical harmonics:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\sum_{\ell m}\frac{1}{2\ell+1}\frac{r'^{\ell}}{r^{\ell+1}}Y^*_{\ell m}(\theta\phi)Y_{\ell m}(\theta\phi) \tag{88}$$

we can write

$$\Phi(\mathbf{r}) = \frac{1}{\epsilon_0}\sum_{\ell m}\frac{1}{2\ell+1}\left[\int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\mathbf{r}')\ d^3\mathbf{r}'\right]\frac{Y_{\ell m}(\theta\phi)}{r^{\ell+1}} \tag{89}$$

By comparison the quantities $q_{\ell m}$ are

$$q_{\ell m} = \int Y^*_{\ell m}(\theta',\phi')r'^{\ell}\rho(\mathbf{r}')\ d^3\mathbf{r}' \tag{90}$$

these coefficients are called multipole moments. monopole term $\ell = 0$:

$$q_{00} = \frac{1}{4\pi}\int\rho(\mathbf{r}')\ d^3\mathbf{r}' = \frac{1}{4\pi}q$$

dipole terms ($\ell = 1$):

$$q_{10} = \sqrt{\frac{3}{4\pi}}\int z'\rho(\mathbf{r}')\ d^3\mathbf{r}' = \sqrt{\frac{3}{4\pi}}p_z$$

and so on.. Eq. 87 can be further recast in

$$\Phi = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\mathbf{x}}{r^3} + \frac{1}{2}\sum_{ij}Q_{ij}\frac{x_ix_j}{r^5} + ...\right] \tag{91}$$

where monopole q, dipole $\mathbf{p}$, and quatrupole Q_{ij} have been introduced.

Multipole expansion of the energy of a charge distribution

The energy of a charge distribution, in an external field, can also be expanded through a multipole expansion.

$$W = \int \rho(\mathbf{r})\Phi(\mathbf{r})\ d^3r \tag{92}$$

Now $\Phi(\mathbf{r})$ can be expandend in a Taylor series around a chosen origin

$$\Phi(\mathbf{r}) = \Phi(0) + \mathbf{r}\cdot\nabla\Phi(0) + \frac{1}{2}\sum_{ij} r_i r_j \frac{\partial^2\Phi}{\partial r_i}\partial r_j(0) + ... \tag{93}$$

this then allows to express the energy as

$$W = q\Phi(0) + \mathbf{p}\cdot\mathbf{E}(0) - \frac{1}{6}\sum_{ij} Q_{ij}\frac{\partial E_j}{r_i}(0) + ... \tag{94}$$

L.8 Laws of magnetostatic

The first part of this lesson is dedicated to the magnetostatic. Laws of magnetostatic, Ampere's law, and Biot-Savart law are given. Definition of solenoid and calculation of the magnetic field inside the solenoid are presented. Next an expression for the force between two wires is derived by starting from the first Ampere's law. Definition of the vector potential $\mathbf{A}$ *(*$\mathbf{B} = \nabla\times\mathbf{A}$*) and Coulomb gauge lead to like-Poisson's equations for* $\mathbf{A}$*. As an application, the potential vector of a current loop is then calculated and verified to satisfy the Biot-Savart law. An expression of the magnetic moment* $\mathbf{m}$ *(localized current distribution) is then derived by a multipole expansion (Jackson eq. 5.51). Examples: calculation of m of a little loop of current; magnetic moment of a charged particle moving in a central field.*

Ampere's first law:

$$\mathbf{dF}(\mathbf{r}) = \frac{I}{c}\mathbf{ds}\times\mathbf{B}(\mathbf{r}) \tag{95}$$

Ampere's law:

$$\oint \mathbf{B}\cdot\mathbf{d}\ell = \mu_0 I \tag{96}$$

Biot-Savart law:

$$\mathbf{dB} = \frac{I}{c}\cdot\frac{\mathbf{ds}\times\mathbf{r}}{r^3} \tag{97}$$

By exploiting the Ampere's first law, the force between two wires can be calculated (Jackson 5.8) as

$$\mathbf{dF}_{12} = \frac{I_1 I_2}{c} \oint_1 \oint_2 \mathbf{ds}_1 \times \mathbf{ds}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{98}$$

now given three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the following vector identity holds: $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, that allows to recast Eq. 99 in:

$$\mathbf{dF}_{12} = \frac{I_1 I_2}{c} \oint_1 \oint_2 \frac{\mathbf{ds}_2[\mathbf{ds}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)] - [\mathbf{ds}_1 \cdot \mathbf{ds}_2](\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{99}$$

The first term is

$$\mathbf{dF}_{12} = \frac{I_1 I_2}{c} \oint_1 \oint_2 \frac{\mathbf{ds}_2[\mathbf{ds}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)]}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \tag{100}$$

$$\oint_2 \mathbf{ds}_2 \oint_1 \mathbf{ds}_1 \cdot \left(-\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) = -\oint_2 \mathbf{ds}_2 \oint_1 d\left(\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) = 0 \tag{101}$$

so that only the second term survives:

$$\mathbf{dF}_{12} = -\frac{I_1 I_2}{c} \oint_1 \oint_2 \frac{[\mathbf{ds}_1 \cdot \mathbf{ds}_2](\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3}. \tag{102}$$

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Let's consider the laws of magnetostatic

$$\nabla \cdot \mathbf{B} = 0 \tag{103}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{104}$$

The second of the Eqs. 103 implies for the vector potential

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J} \tag{105}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{106}$$

Now since $\mathbf{A}$ is not completely fixed by the definition $\mathbf{B} = \nabla \times \mathbf{A}$ we can make a choice: $\nabla \cdot \mathbf{A} = 0$ known as "Coulomb gauge". This leads to the Poisson-like equation

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{107}$$

whose general solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' + \nabla\Lambda(\mathbf{r}). \tag{108}$$

Now, since we have the "Coulomb gauge" $\nabla \cdot \mathbf{A} = 0$, it follows $\nabla \cdot \nabla\Lambda = 0$ i.e $\nabla^2\Lambda(\mathbf{r}) = 0$ for each $\mathbf{r}$ point in the space. This means $\Lambda = 0$.

Let's consider a loop with current I. What is the generated $\mathbf{B}$ field ? Let ds be a small enough path along the loop. Then $\mathbf{J}(\mathbf{r}')d^3\mathbf{r}' = JS\ ds$, where S is the section of the wire and $JS = I$. We can thus write the vector potential as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{I}{|\mathbf{r} - \mathbf{r}'|} \mathbf{ds} \tag{109}$$

and then it follows

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \oint I \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^2} \mathbf{ds_2}. \tag{110}$$

The magnetic potential energy can be expressed as:

$$U = \frac{1}{2} \int \mathbf{J} \cdot \mathbf{A}\ d^3r \tag{111}$$

L.9 Exercises

A problem of finding the potential for a grounded sphere embedded in a region with a non-zero electric field is presented and solved through the method of image charges . Next follows a problem concerning polynomials in the Hilber space L2(-1,1). Polynomials form an orthogonal basis in such a space. Moreover by a Gram-Schmidt procedure it is possible to find an orthonormal basis set: the Legendre polynomials . Similarly, the set $e^{im\phi}$ is shown to form an orthonormal basis set of L2(0,2π) once proper normalization coefficients are found. Finally, it is demonstrated how to calculate the electric quadrupole moment for a configuration of four charges ("double dipole").

L.10 Faraday's law and Conservation laws

Faraday's law of induction is presented in both differential and integral forms. Next, the equation of continuity for the electric charge is discussed as an example of local conservation law. It is then shown how Maxwell had to introduce a new term, the displacement current ($\partial\mathbf{E}/\partial t$) in order to satisfy

the charge continuity equation. This term allows Maxwell's equation to produce electromagnetic waves. Next an exercise is proposed: find the current in a rotating circuit in a region of constant magnetic field. The situation of two entangled solenoids is then exploited to introduce the Lenz law ("a manifestation of the energy conservation") and the concept of mutual inductance. The lesson proceeds with conservation of energy and momentum: the Poynting's Theorem [Jackson, sec. 6.7]. The meaning of the Poyinting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ *is discussed with a few examples (e.g plane capacitor).*

The law of induction (Faraday's law) is:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{112}$$

equivalently

$$\mathcal{E}_{MF} = \oint \mathbf{E} \cdot \mathbf{d}\ell = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{dS} \tag{113}$$

The equation of continuity for the electric charge is:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{114}$$

and represents a *local conservation.*

Maxwell had to add the displacement current $\frac{\partial \mathbf{E}}{\partial t}$ in

$$c^2 \nabla \times \mathbf{B} = \frac{1}{\epsilon_0} \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{115}$$

because otherwise this equation would always give $\nabla \cdot \mathbf{J} = 0$ in contrast with the continuity equation. He also found that $\sqrt{\epsilon_0 \mu_0} = \frac{1}{c}$.

Power:

$$P = \int \mathbf{J} \cdot \mathbf{E} \; d^3V \tag{116}$$

The former equation can be thought as a generalization of the power definition in mechanics as $\mathbf{F} \cdot \mathbf{v}$. Note also tha $\mathbf{B}$ does not contribute because the force of the magnetic field is orthogonal to both $\mathbf{v}$ and $\mathbf{B}$. From Maxwell equation Eq. 115 we know that

$$\mathbf{J} = \epsilon_0 c^2 (\nabla \times \mathbf{B}) - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{117}$$

By exploiting the definition of energy density

$$u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) \tag{118}$$

and

$$\mathbf{E}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{B}) \tag{119}$$

we can write for the power of the electromagnetic field:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{dt}} = \int \mathbf{J}\cdot\mathbf{E}\; d^3V = -\int d^3V\left[\frac{\mathrm{d}u}{\mathrm{dt}} + \nabla\cdot(\mathbf{E}\times\mathbf{B})\right]. \tag{120}$$

This leads to the introduction of a new vector field $\mathbf{S} = \mathbf{E}\times\mathbf{B}$, the Poynting vector, that tells us how the fields move so that the energy is conserved. Since the volume V is arbitrary in Eq. 120, this has to be true for all volumes, implying that

$$-\frac{\mathrm{d}u}{\mathrm{dt}} = \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E}. \tag{121}$$

which is Poynting's theorem in differential form.

Example: plane capacitor

Let's consider a plane capacitor with area $A = \pi a^2$ and height h. Then $U = \frac{\epsilon_0}{2}E^2\pi a^2 h$ and

$$\frac{dU}{dt} = \epsilon_0\pi a^2 h E\dot{E}.$$

Moreover from the equation $c^2\nabla\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t}$ we obtain

$$2\pi a c^2 B = \dot{E}\pi a^2$$

i.e

$$B = \dot{E}a/2c^2.$$

The Poynting vector is

$$S = \epsilon_0 c^2\mathbf{E}\times\mathbf{B} = \frac{\epsilon_0}{2}aE\dot{E}$$

and the flux of S through the lateral area $2\pi a h$ is

$$\frac{\epsilon_0}{2}aE\dot{E}\cdot 2\pi a h = \epsilon_0\pi a^2 h E\dot{E}.$$

Conservation of momentum

Maxwell stress tensor $T_{\alpha\beta}$

$$\frac{d}{dt}(P_{mech} + P_{fields})_\alpha = \int_{surf}\sum_\beta T_{\alpha\beta}\eta_\beta d\sigma \tag{122}$$

(to be updated)

L.11 The wave equation

The first part of this lesson concerns the definitions of vector and scalar potential. Maxwell equations are rewritten in terms of these quantities. The concept of gauge-invariance is introduced and the Lorentz gauge is discussed [Jackson sect. 6.2 - 6.3]. The second part of this lesson is devoted to the Green functions of the wave equation (see Jackson sect. 6.4). Definitions and meaning of the retarded and advanced Green functions are discussed. Complete solutions of the inhomogeneous wave equation are given in terms of the retarded and advanced Green functions.

By using the definition of vector potential we can rewrite the Faraday's law as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{123}$$

This we now means that a potential Φ exists such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi$$

i.e.

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Let's consider the inhomogeneous Maxwell equation in vacuum and write them in terms of the potentials ϕ, $\mathbf{A}$. We thus find the perfectly equivalent equations:

$$\nabla^2 \Phi + \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t} = -\frac{\rho}{\epsilon_0} \tag{124}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla (\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \Phi}{\partial t}) = -\mu_0 \mathbf{J}. \tag{125}$$

Now, (gauge invariance of the potentials), we note that the potentials can be transformed by means of an arbitrary function Λ, $\mathbf{A}' = \mathbf{A} + \nabla \Lambda$ and $\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$ without changing the electric and magnetic fields, e.g:

$$\mathbf{E}' = -\nabla (\Phi - \frac{\partial \Lambda}{\partial t}) - \frac{\partial (\mathbf{A} + \nabla \Lambda)}{\partial t} = \mathbf{E}. \tag{126}$$

This allow us to choose ϕ, $\mathbf{A}$ such that the following condition [Lorenz condition (gauge)] is satisfied:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0. \tag{127}$$

By doing that we find the inhomogeneous wave equations of the potentials:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{128}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{129}$$

The former equations share the general form of the he inhomogeneous wave equation:

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(\mathbf{r},t). \tag{130}$$

By Fourier transforming we obtain the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\psi(\mathbf{r},\omega) = -4\pi f(\mathbf{r},\omega). \tag{131}$$

where $k = \omega/c$. This is an elliptic partial differential equation that becomes the Poisson equation when $k = 0$. Let's consider the associated Green's function

$$(\nabla^2 + k^2)G_k(\mathbf{r},\mathbf{r}') = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{132}$$

Let's put $R = |\mathbf{r}-\mathbf{r}'|$ we then obtain (assume no boundary surfaces)

$$\frac{1}{R}\frac{\partial^2 RG_k}{\partial R^2} + k^2 G_k = -4\pi\delta(R) \tag{133}$$

that has solution

$$RG_k(R) = Ae^{ik\cdot R} + Be^{-ik\cdot R} \tag{134}$$

in other words the solutions are spherical waves

$$G_k^{\pm}(R) = \frac{e^{\pm ikR}}{R} \tag{135}$$

coming back to the time domain

$$G^{\pm}(R,t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ikR}}{R}e^{-i\omega(t-t')}\,d\omega. \tag{136}$$

By remembering the definition of delta function we then obtain

$$G^{\pm}(R,t-t') = \frac{1}{R}\delta(t-t'\mp\frac{R}{c}). \tag{137}$$

Now if we expand $R = |\mathbf{r}-\mathbf{r}'|$ in the previous equation we can write

$$G^{\pm}(R,t-t') = \frac{\delta(t' - [t \mp \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|} \tag{138}$$

where $t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$ is the retarded time, while $t' = t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}$ is called advanced time. Correspondingli G^+ is the retarded Green function and G^- the advanced one. The full solution of the wave equation is then obtained as

$$\psi(\mathbf{r},t) = \psi_{in}(\mathbf{r},t) + \int d^3\mathbf{r}'dt' \; G^+(\mathbf{r}t,\mathbf{r}'t')f(\mathbf{r}'t') \tag{139}$$

Remark
When calculating $\mathbf{A}$ and Φ one has to compute the charge as it was "before", $t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$, because what ever the charge is doing it will reach the observer only at a later time:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \; \frac{[\mathbf{J}(\mathbf{r}',t')]_{ret}}{|\mathbf{r}-\mathbf{r}'|} \tag{140}$$

$$\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3\mathbf{r}' \; \frac{[\rho(\mathbf{r}',t')]_{ret}}{|\mathbf{r}-\mathbf{r}'|} \tag{141}$$

where ret means that we have to compute ρ, $\mathbf{J}$ at $t' = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$.

L.12 Exercises

In the next two hours, students show the solution of five homeworks they were given in the previous lessons. The first exercise consist in finding the expression of the force between two infinite long wires carrying currents I_1 and I_2 . The second task consists in deriving the expression of the B field given a vector potential of the form

$$\mathbf{A} = (\mathbf{m}\times\mathbf{r})/r^3. \tag{142}$$

The third problem consist in finding the current flowing in a rotating circuit (a square of area a) in a constant B field region. In the next exercise, the differential equation for the current in a RLC circuit is written and solved. The last problem consist in showing that Maxwell's equations as well as the Poynting vector and the energy density are invariant under duality transformations [Jackson, sec. 6.11, p. 274].

Exercise 1

The magnetic field generated by a wire with current I is

$$B = \frac{\mu_0 I}{2\pi d}$$

at a distance d (SI units). By using Ampere's law

$$d\mathbf{F} = Id\mathbf{s} \times \mathbf{B}$$

we can find the force (per unit length) between two infinite long wires

$$\mathbf{F} = \frac{\mu_0}{2\imath d} I_1 I_2$$

Exercise 2

$$\mathbf{A} = \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$

we can compute the magnetic field by applying the definition $\mathbf{B} = \nabla \times \mathbf{A}$. The result is

$$\mathbf{B} = \frac{\mathbf{e}_r(\mathbf{e}_r \cdot \mathbf{m}) - \mathbf{m})}{r^3}$$

where $\mathbf{e}_r = \mathbf{r}/r$.

L.13 Radiation from an oscillating charge distribution

In the begining of this lesson, corrections to the potentials Φ and $\mathbf{A}$ are derived leading to the so-called retarded potentials (Liénard-Wiechert potentials) where it is taken into account that a signal takes a finite time, corresponding to the velocity of light, to propagate from the source point of the field to the observation point $\mathbf{r}$. Next, the problem of the radiation from an oscillating charge distribution is considered. Solutions for the vector potential are found for both the near zone and far zone. For the latter it is shown that the leading term of the vector potential depends on the electric dipole moment $\mathbf{p}$. Expressions for the fields $\mathbf{E}, \mathbf{H}$ and for the power radiated per unit solid angle are then given [Jackson, sec. 14.3]. In the last part of the lesson it is shown how by starting from the definition of dipole and by using the Liénard-Wiechert potentials it is possible to come to the result previously found for the radiated power (Hertz oscillator).

Let's find the corrections to the Eqs. 140

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \, \frac{\rho(\mathbf{r}', t')}{R}|_{ret} \tag{143}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3\mathbf{r}' \, \frac{\mathbf{J}(\mathbf{r}', t')}{R}|_{ret} \tag{144}$$

for a source moving with velocity $\mathbf{v}$ ($v << c$) and where $t' = t - \frac{R}{c}$ and $R = |\mathbf{r} - \mathbf{r}'|$.

Now

$$dq = \rho(\mathbf{r}', t')d^3\mathbf{r}' - \rho(\mathbf{r}', t')\frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{v}}{|\mathbf{r} - \mathbf{r}'|c}d^3\mathbf{r}' = \rho(\mathbf{r}', t')\left(1 - \frac{(\mathbf{r} - \mathbf{r}') \cdot \mathbf{v}}{|\mathbf{r} - \mathbf{r}'|c}\right)d^3\mathbf{r}'$$

then we have

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\mathbf{r} - \mathbf{r}'|} = \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}\frac{1}{(1 - v/c)|_{ret}} \tag{145}$$

Similarly one can find the corrected expression for the vector potential:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{|\mathbf{r} - \mathbf{r}'|}\frac{1}{(1 - v/c)|_{ret}} \tag{146}$$

The previous corrections to the potentials are known as "Liénard-Wichert potentials".

Oscillating source

Let's consider an oscillating source

$$\rho(\mathbf{r}, t) = \rho(\mathbf{r})e^{-i\omega t} \tag{147}$$

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}, \tag{148}$$

From these we can obtain $\mathbf{E}$ and $\mathbf{B}$. Indeed

$$\nabla \times \mathbf{E} - i\omega\mathbf{B} = 0 \tag{149}$$

$$\nabla \times \mathbf{H} + i\omega\mathbf{D} = 0 \tag{150}$$

Let's calculate the vector potential

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \int dt' \frac{\mathbf{J}(\mathbf{r}'t')}{|\mathbf{r} - \mathbf{r}'|}\delta(t' + \frac{|\mathbf{r} - \mathbf{r}'|}{c} - t). \tag{151}$$

If we now explicit the time dependence of the current $\mathbf{J}$ and we perform the integral on t' we obtain:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}e^{ik|\mathbf{r}-\mathbf{r}'|-i\omega t}, \tag{152}$$

where $k = \omega/c$. Now the solution can take a different form according how d, r, and λ are ordered where d is the dimension of the source, r distance source-observer, and $\lambda = \frac{2\pi c}{\omega}$ is the characteristic length of this oscillating source.

We then have three regions: i) near zone: $d << r << \lambda$ ii) intermediate zone $d << r \sim \lambda$ iii) far zone $d << \lambda << r$. The study of the ii) zone goes beyond the scope of this course therefore we will not treat it here. **(i) near zone**: $d << r << \lambda$. In this region $kr << 1$ and $e^{ik|\mathbf{r}-\mathbf{r}'|} \to 1$.

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{Y_{\ell m}(\theta\phi)}{r^{\ell+1}}e^{-i\omega t}\int d^3\mathbf{r}'\mathbf{J}(\mathbf{r}')r'^{\ell}Y^*_{\ell m}(\theta'\phi') \tag{153}$$

In the previous equation the time dependence enters only through the factor $e^{-i\omega t}$ but except this all the rest is as in the static solution.
(iii) far zone: $d << \lambda << r$. In this region we can exploit the following approximation:

$$|\mathbf{r}-\mathbf{r}'| \simeq \mathbf{r} - \frac{\mathbf{r}}{r}\cdot\mathbf{r}' = \mathbf{r} - \mathbf{n}\cdot\mathbf{r}'. \tag{154}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{e^{ikr-i\omega t}}{r}\int \mathbf{J}(\mathbf{r}')e^{-ik\mathbf{n}\cdot\mathbf{r}'}d^3\mathbf{r}' \tag{155}$$

The former equation describes a propagating spherical wave. Let's now derive a multipole expansion. Let's express the exponential in the integral as a power series:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{e^{ikr-i\omega t}}{r}\sum_n\frac{(-ik)^n}{n!}\int \mathbf{J}(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')^n d^3\mathbf{r}'. \tag{156}$$

Let's now focus on the first term of this expansion

$$\mathbf{A}^1(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{e^{ikr-i\omega t}}{r}\sum_n\int \mathbf{J}(\mathbf{r}')d^3\mathbf{r}'. \tag{157}$$

By integrating by part and by taking into account that r' is small wrt r we obtain

$$\int \mathbf{J}(\mathbf{r}')d^3\mathbf{r}' = -\int \mathbf{r}'\cdot\nabla\cdot\mathbf{J}(\mathbf{r}')d^3\mathbf{r}' = -i\omega\int \mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}'. \tag{158}$$

The latter gives the electric dipole moment $\mathbf{p}$ and thus we can write the leading term of the radiative field $\mathbf{A}$ as

$$\mathbf{A}^1(\mathbf{r},t) = -i\frac{\mu_0\omega}{4\pi}\frac{e^{ikr-i\omega t}}{r}\mathbf{p} \tag{159}$$

Let's find the fields $\mathbf{E}$, and $\mathbf{H}$:

$$\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} = \frac{ck^2}{4\pi}(\mathbf{n}\times\mathbf{p})\frac{e^{ikr-i\omega t}}{r} \tag{160}$$

$$\mathbf{E} = \mathbf{H}\times\mathbf{n}\sqrt{\frac{\mu_0}{\epsilon_0}}. \tag{161}$$

Let's calculate the total power.

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2\mathbf{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \frac{c^2k^4}{32\pi^2}\sqrt{\frac{\mu_0}{\epsilon_0}}|\mathbf{p}|^2\sin^2\theta \tag{162}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{r}$. The total power is then:

$$P_{tot} = \int d\Omega\frac{dP}{d\Omega} = \frac{c^2k^4}{12\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}|\mathbf{p}|^2 \tag{163}$$

Hertz oscillator

Let's consider an oscillating dipole $\mathbf{p} = e\mathbf{r}'$ where two charges e and $-e$ are placed at distance $\mathbf{r}'/2$ and $-\mathbf{r}'/2$ from the origin. Eqs. 145 and 146 become

$$\Phi(\mathbf{r},t) = \frac{e}{|\mathbf{r}-\mathbf{r}'| - \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\cdot\frac{\mathbf{v}}{c}}|_{ret} \tag{164}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\frac{e}{c}\mathbf{v}}{|\mathbf{r}-\mathbf{r}'| - \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\cdot\frac{\mathbf{v}}{c}}|_{ret} \tag{165}$$

We now define $\mathbf{r}_\mp = \mathbf{r}\mp\mathbf{r}'/2$. Now let's apply the previous definitions (i.e. the Liénard Wichert potentials) for both the charges of the dipole:

$$\Phi = \frac{-e}{|\mathbf{r}_-| - \frac{\mathbf{r}_-}{|\mathbf{r}_-|}\cdot\frac{\mathbf{r}'}{2c}}|_{ret} + \frac{e}{|\mathbf{r}_+| - \frac{\mathbf{r}_+}{|\mathbf{r}_+|}\cdot\frac{-\mathbf{r}'}{2c}}|_{ret} \tag{166}$$

$$\mathbf{A} = \frac{-e\dot{\mathbf{r}'}/2c}{|\mathbf{r}_-| - \frac{\mathbf{r}_-}{|\mathbf{r}_-|}\cdot\frac{\mathbf{r}'}{2c}}|_{ret} + \frac{e(-\dot{\mathbf{r}'})/2c}{|\mathbf{r}_+| - \frac{\mathbf{r}_+}{|\mathbf{r}_+|}\cdot\frac{-\mathbf{r}'}{2c}}|_{ret} \tag{167}$$

$$\tag{168}$$

Usually we are allowed (the typical situation) to apply the following approximations $r_\pm = r$ and $v << c$. This lead to the following expression for the vector potential:

$$\mathbf{A}(\mathbf{r},t) \approx -\frac{e}{c}\frac{\dot{\mathbf{r}'}}{|\mathbf{r}|} = \frac{\dot{\mathbf{p}}}{cr}|_{ret}. \tag{169}$$

From this expression we can find the fields $\mathbf{H}$ and $\mathbf{E}$:

$$\mathbf{H} = \frac{\ddot{\mathbf{p}}\times\mathbf{r}}{c^2r^2} \tag{170}$$

$$\mathbf{E} = \frac{1}{c^2r^3}(\ddot{\mathbf{p}}\times\mathbf{r})\times\mathbf{r} \tag{171}$$

Now let's explicitly write the retarded time in the expression of the vector potential:

$$\frac{\dot{\mathbf{p}}}{cr}|_{ret} = \frac{\frac{d}{dt}(\mathbf{p}_0 e^{-i\omega t'})}{cr}|_{t'=t-\frac{|\mathbf{r}-\mathbf{r}'}{c}=t-r/c} = -i\omega \mathbf{p}_0 \frac{e^{ikr-i\omega t}}{cr} \tag{172}$$

This expression looks like Eq. 159 and thus it follows that the expressions for the radiated power are the Eqs. 162 and 163. Notice that there is no radiation in the direction of the motion of the charges ("synchrotron radiation").

Exercise

Given the Liénard-Wichert potentials Eqs. 145 and 146 determine the **E** and **B** fields.

L.14 Maxwell equations in the absence of sources

In this lecture it is shown how the Maxwell equations in the absence of sources lead to wave equations for the electric and magnetic fields (Helmholtz equations). Moreover these fields are mutually orthogonal and propagate at the speed of light. Definitions of linear, circular and elliptic polarization are given. The Poynting vector of electromagnetic waves is shown to be along the direction of propagation and its modulus is proportional to the energy density. Refraction and reflection of electromagnetic waves at the interface between two media (e.g water/air) is then discussed. Fresnel equations and Snell's law are derived by exploiting Maxwell's equations. Next, non-dispersive and dispersive media are introduced and definitions of group velocity and phase velocity are given.

Let's consider the Maxwell equations in the absence of sources

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \qquad & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{B} = 0 \qquad & c^2 \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0 \end{aligned} \tag{173}$$

If we consider a $e^{-i\omega t}$ time dependence for the fields **B**, and **E** we can write

$$\nabla \times \mathbf{E} = i\omega B \tag{174}$$

$$\nabla \times \mathbf{B} = -i\frac{\omega}{c^2}\mathbf{E}. \tag{175}$$

Now if we take the curl of the first equation

$$\nabla \times (\nabla \times \mathbf{E}) - i\omega \nabla \times B = 0$$

that thanks to Eq. 175 becomes

$$\nabla \cdot (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - i\omega(-i\frac{\omega}{c^2}\mathbf{E}) = 0 \tag{176}$$

and since $\nabla \cdot \mathbf{E} = 0$ we obtain

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = 0 \tag{177}$$

and similarly for the $\mathbf{B}$ field:

$$\nabla^2 \mathbf{B} + \frac{\omega^2}{c^2}\mathbf{B} = 0. \tag{178}$$

The former are known as Helmholtz wave equations and have solutions of the form:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{A}_E e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{179}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{A}_B e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{180}$$

$$\tag{181}$$

where $\mathbf{k}$ gives the direction of propagation and $k = \omega/c$. Now the left side of Eqs. 173 imply that $\mathbf{A}_E \cdot \mathbf{k} = 0 = \mathbf{k} \cdot \mathbf{A}_B$ i.e. the wave amplitude is orthogonal to the direction of propagation (trasversal wave). Moreover from the right side of Eqs. 173 we can see that

$$\mathbf{A}_B = \frac{\mathbf{k} \times \mathbf{A}_E}{\omega}$$

that leads to:

$$|\mathbf{E}| = c|\mathbf{B}|. \tag{182}$$

The fact that the oscillation is orthogonal to the direction of propagation $\mathbf{k}/|\mathbf{k}|$ implies that only two polarization vectors $\mathcal{E}_1$, $\mathcal{E}_2$ are required to specify the field amplitude:

$$\mathbf{E} = (E_1\mathcal{E}_1 + E_2\mathcal{E}_2)e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{183}$$

Polarization can be defined as linear if the direction of the electric field is constant $\theta = arctan\frac{E_2}{E_1}$ and $E = \sqrt{E_1^2 + E_2^2}$. Now if $E_2 = E_1 e^{i\phi}$ and if $\phi = \pm\frac{\pi}{2}$ we have circular polarization

$$\mathbf{E} = E_1(\mathcal{E}_1 \pm i\mathcal{E}_2)e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} \tag{184}$$

and for a generic ϕ and $E_1 \neq E_2$ we have elliptic polarization. Note that the natural light from the Sun is not polarized. Light polarization phenomena cang occur e.g. after reflection or refraction at the interface between two media.

Let's calculate the Poynting vector of the wave:

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{B} = \frac{1}{2}\mathbf{E} \times \frac{(\mathbf{k} \times \mathbf{E})}{\omega} = \frac{\mathbf{k}}{2ck}|\mathbf{E}|^2 = \frac{u}{c}\hat{k} \tag{185}$$

where $u = |\mathbf{E}|^2/2$. Thus the Poynting vector of electromagnetic waves is along the direction of propagation and its modulus is proportional to the energy density.

Reflection and refraction of waves

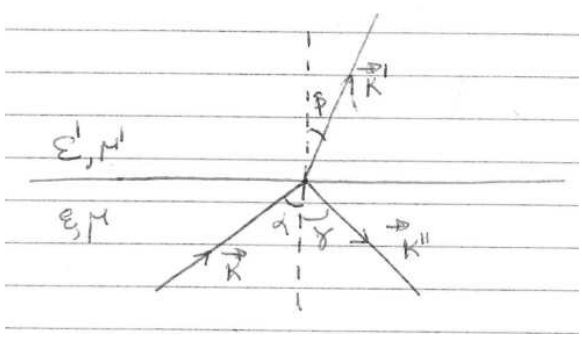
[to be updated]

Snell's law:

$$\frac{\sin\alpha}{\sin\beta} = \frac{k'}{k} = \frac{n'}{n} \tag{186}$$

where n and n' are the refraction index of the incoming light medium and outcoming light medium.

Figure L.5: *Light refraction at the interface between two media*



Media can be non dispersive if $d\epsilon/d\omega = 0$ e.g. air or they can be dispersive if $\epsilon = \epsilon(\omega)$ and $\omega = \omega(k)$. The general solution of the wave equation

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx - i\omega(k)t} dk \tag{187}$$

is given by a superposition of monochromatic waves.

Now

$$\omega(k) = \omega_0 + \frac{d\omega}{dk}|_{k=k_0}(k - k_0) + ... \tag{188}$$

and

$$v_g = \frac{d\omega}{dk}|_{k=k_0} \tag{189}$$

is called *group velocity*, while

$$v = \omega(k)/k = c/n(k)$$

is called *phase velocity*.

L.15 Exercises

In this lecture, the solutions of a few homeworks are illustrated by the students. The first exercise concerns the derivation of the electric and magnetic fields for a moving point charge q by using as starting point the expressions given in the previous lessons of the Liénard-Wiechert potentials. The second exercise consists in the calculation of the total radiated power for a charge moving along a circular trajectory at a constant speed (Larmor formula, see Jackson sec. 14.2). The following exercise shows how to write the differential equation for a transmission line represented by a discrete ladder network of conductors and capacitors.

Electric and magnetic fields for a moving charge q

Starting from the retarded potentials

$$\phi = \frac{e}{R\ (1 - \beta \cdot \mathbf{n})}|_{ret} \tag{190}$$

$$\mathbf{A} = \frac{\mathbf{v}}{c}\phi|_{ret} \tag{191}$$

we want to derive $\mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$ and $\mathbf{H} = \nabla \times \mathbf{A}$. Here $R = c(t - t')$ and $t' = t - R(t')/c$ is the retarded time. Moreover $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, $\mathbf{n} = \mathbf{R}/R$, and $\beta = \mathbf{v}/c$.

Suggestions:

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \beta \cdot \mathbf{n}} \tag{192}$$

and

$$\nabla t' = -\frac{\mathbf{R}}{c(R - \frac{\mathbf{R}\cdot\mathbf{v}}{c}).} \tag{193}$$

Moreover (chain rule also useful for a generic function):

$$\nabla R(t') = (\nabla R)|_{ret} + \frac{\partial R}{\partial t'}\nabla t' = \frac{\mathbf{R}}{R} + \frac{\partial R}{\partial t'}\nabla t' \tag{194}$$

The result is:

$$\mathbf{E} = e\frac{1 - v^2/c^2}{(R - \frac{\mathbf{R}\cdot\mathbf{v}}{c})^3}(\mathbf{R} - \frac{\mathbf{v}}{c}R) + \frac{e}{c^2(R - \frac{\mathbf{R}\cdot\mathbf{v}}{c})^3}\mathbf{R} \times [(\mathbf{R} - \frac{\mathbf{v}}{c}R) \times \dot{\mathbf{v}}] \tag{195}$$

where $\dot{\mathbf{v}} = \partial\mathbf{v}/\partial t'$ and

$$\mathbf{H} = \frac{1}{R}\mathbf{R} \times \mathbf{E} \tag{196}$$

Note that $\mathbf{H}$ field is always orthogonal to the $\mathbf{E}$ field. [See Landau-Lifshitz's Field Theory]

Exercise: the Larmor formula

Calculate the radiation of a rotating charge

$$\mathbf{r}(t) = a[\mathbf{e}_x \cos\omega_0 t + \mathbf{e}_y \sin\omega_0 t]$$

with dipole $\mathbf{d} = e\mathbf{r}(t)$. The radiated power can be written as

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}|\dot{\mathbf{v}}|^2 \sin^2\theta \tag{197}$$

Exercise

Let's write the differential equation for the potential V along the transmission line:

$$V_n - V_{n-1} = V_L = -L\frac{dI_{n-1}}{dt}$$

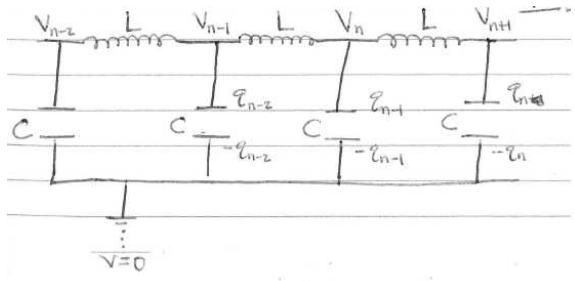
dividing by $\delta x = x_n - x_{n-1}$ one obtains:

$$\frac{V_n - V_{n-1}}{\delta x} = -\frac{L}{\delta x}\frac{dI_{n-1}}{dt}$$

In the limit of small δx

$$\frac{\partial V}{\partial x} = -L_0\frac{\partial I}{\partial t} \tag{198}$$

Figure L.6: *transmission line represented by a discrete ladder network of conductors and capacitors.*



where $L_0 = L/\delta x$. Then

$$I_{n-1} = I_n + I_C$$

$$I_n - I_{n-1} = -I_C = -C\frac{\partial V}{\partial t}$$

and by dividing by δx

$$\frac{I_n - I_{n-1}}{\delta x} = -\frac{C}{\delta x}\frac{\partial V}{\partial t}$$

In the limit of small δx

$$\frac{\partial I}{\partial x} = -C_0\frac{\partial V}{\partial t} \tag{199}$$

where $C_0 = C/\delta x$. By deriving Eq. 198 by $\partial/\partial x$ we obtain

$$\frac{\partial^2 V}{\partial x^2} = -L_0\frac{\partial^2 I}{\partial x \partial t} = C_0 L_0 \frac{\partial^2 V}{\partial t^2}.$$

This is a wave equation, with wave velocity $v = 1/\sqrt{L_0 C_0}$. A similar equation can be derived for the current, I.

L.16 Lorentz's transformations

While Newton's law is invariant under Galileian transformations, the wave equation is not as shown in the first part of this lesson. The transformations that preserve the wave equation are then derived by imposing the condition to leave unchanged the wave front i.e. the quantity $x^2 + y^2 + z^2 - (ct)^2$ [Jackson, sec. 11.3]. These transformations are called "Lorentz's transformations" and in particular, the boost transformations are here discussed.

The concept of Minkowski metric is introduced and discussed with the help of a two dimensional space-time model. Next, twins paradox is illustrated as an example of time-dilation. Similarly the Lorentz contraction is discussed and examples are mentioned (e.g. ladder paradox).

Let's consider the wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\psi(\mathbf{x}, t) = 0 \tag{200}$$

and let's see how this equation transforms under a Galileian transformation

$$x' = x - vt \tag{201}$$
$$t' = t \tag{202}$$

this implies:

$$\frac{d\psi}{dx} = \frac{d\psi}{dx'} \tag{203}$$
$$\frac{d\psi}{dt} = -v\frac{d\psi}{dx'} + \frac{d\psi}{dt'} \tag{204}$$

In the full 3D formalism the former equation becomes

$$\frac{d\psi}{dt} = -\mathbf{v}\cdot\nabla\psi + \frac{d\psi}{dt'} \tag{205}$$

and the wave equation reads:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2}{c^2}\mathbf{v}\cdot\nabla\frac{\partial}{\partial t} - \frac{1}{c^2}(\mathbf{v}\cdot\nabla)(\mathbf{v}\cdot\nabla)\right)\psi(\mathbf{x}', t) = 0 \tag{206}$$

So the wave equation is not invariant under Galileian transformation (Other equations may be invariant for instance Schrödinger equation). What are the transformations that leave invariant the wave equation? Let's look for those that satisfies

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2. \tag{207}$$

Let's define $x^0 = ct$ and $x^1 = x$, $x^2 = y$, $x^3 = z$. Then

$$\sum_\mu x_\mu^2 = \sum_\mu x_\mu'^2 \tag{208}$$

This lead to the linear transformations $a_{\mu\nu}$

$$x'_\mu = \sum_\nu a_{\mu\nu} x^\nu \tag{209}$$

with (orthogonal transformation)

$$\sum_\nu a^\nu_\mu a^\lambda_\nu = \delta^\lambda_\mu. \tag{210}$$

For a transformation in the x^1 direction we can derive the following Lorentz's transformation (boost):

$$x' = \gamma(x - vt) \tag{211}$$

$$t' = \gamma(t - \frac{v}{c^2}x) \tag{212}$$

$$y' = y \tag{213}$$

$$x' = z \tag{214}$$

L.17 Covariance of the Maxwell's equations

In this lessons the definition of four-vectors and of scalar product between four-vectors is given and this leads to the Minkowski four-dimensional space whose metric is

$$ds^2 = \eta_{ab} dx_a dx_b = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{215}$$

In particular, it is given the definition of the 4-velocity ua and of the four-momentum pa. Next, a few homeworks are suggested to students, e.g. demonstrate that

$$E^2 = c^2(m^2c^2 + p^2). \tag{216}$$

Next the acceleration four-vector is introduced by discussing an example: a rocket with g acceleration. In particular, the solution of the uniformly accelerated motion is found. Next, the Lorentz's force is written in a covariant form by introducing a 4-tensor $F_{\alpha\beta}$, i.e. the electromagnetic 4-tensor. The components of this antisymmetric tensor are given by the components of the electric and magnetic fields. This allows us to write Maxwell's equations in a very compact form [Jackson sec. 11.9]. Moreover this result together with the definitions of the 4-vectors J_α and A_α establishes the covariance of the equations of electromagnetism.

The Lorentz's force that we have seen at the beginning of these notes:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \tag{217}$$

can be written in a covariant form:

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} v_\beta \tag{218}$$

where $F^{\alpha\beta}$ are the components of the electromagnetic 4-tensor:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & -B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \tag{219}$$

Maxwell's equations in a covariant form

Let's define $J^\alpha = (c\rho, \mathbf{J})$ and $\partial^\alpha = \frac{\partial}{\partial x^\alpha} = (-\frac{\partial}{\partial x^0}, \nabla)$, then the continuity equation can be written as

$$\partial^\alpha J^\alpha = 0. \tag{220}$$

Similarly, Gauss and Ampere's law can be written in the compact form (Gaussian units):

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\alpha. \tag{221}$$

The other equations (in the absence of charges and currents) become:

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0. \tag{222}$$

Exercise

Let $A^\alpha = (\phi, \mathbf{A})$ be a "quadri-potential" (4-vector) and let's apply a boost (see Lorentz's transformations above) for a moving charge with a constant speed v along x direction (Fig. L.8). Show that after the transformation the Liénard-Wichert potentials are recovered.

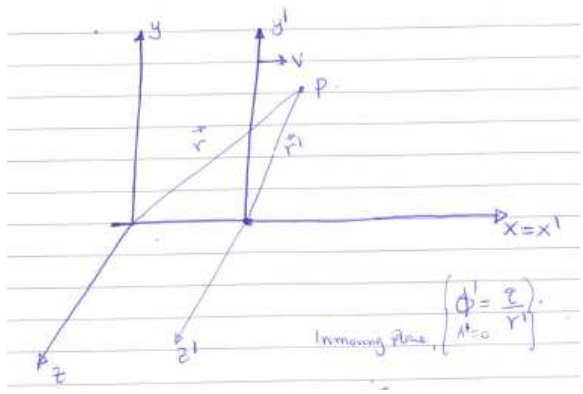
By applying the boost in the x-direction we have

$$\phi' = \frac{\phi - vA_x}{\sqrt{1-v^2}} \tag{223}$$

and

$$A'_x = \frac{A_x - v\phi}{\sqrt{1-v^2}} \tag{224}$$

Figure L.7: *Boost transformation with a constant speed v along x direction*



Now for a charge at rest in the prime reference system $A' = 0$ and $\phi' = \frac{q}{r'}$. It follows ($c = 1$)

$$\phi = \frac{\phi' + vA'_x}{\sqrt{1-v^2}} = \frac{q}{[\sqrt{1-v^2}]r'} = \frac{q}{\sqrt{1-v^2}\left[(\frac{x-vt}{\sqrt{1-v^2}})^2 + y^2 + z^2\right]^{1/2}} \tag{225}$$

Moreover

$$A_x = v\phi \tag{226}$$

The previous equations are the Liénard-Wichert potentials (See Eqs. 145 and 146).

L.18 Exercises

The first exercise of this lecture concerns the electromagnetic waves in a uniform gas of free electrons having density ne. By applying Maxwell's equations a dispersion relation is derived where a cutoff frequency can be defined (plasma frequency). For large enough ne it is shown that the wave is absorbed. The next exercise consists in an example of the twin paradox. The concept of simultaneity is then discussed. The following problem treats the problem of the emission or absorption of light by a free electron. This turns out to be impossible.

Electromagnetic waves in a uniform gas of free electrons

Let's consider a EM wave

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}. \tag{227}$$

Now for a charge $-e$ we have

$$\mathbf{F} = m\dot{\mathbf{v}} = -e\mathbf{E}$$

that leads to

$$\mathbf{v} = \frac{e\mathbf{E}}{i\omega m}. \tag{228}$$

Now the current density will be

$$\mathbf{J} = -en_e\mathbf{v} = i\frac{e^2 n_e}{m\omega}\mathbf{E}. \tag{229}$$

By Maxwell equations (See L.2 and L.14) we then obtain

$$-\nabla^2\mathbf{E} = \left(\frac{\mu_0 e^2 n_e}{m\omega^2} - \frac{1}{c^2}\right)\frac{\partial^2}{\partial t^2}\mathbf{E} \tag{230}$$

This for Eq. 227 means:

$$\mathbf{k}^2 + \left(\frac{\mu_0 e^2 n_e}{m\omega^2} - \frac{1}{c^2}\right)\omega^2 = 0 \tag{231}$$

and thus

$$c^2\mathbf{k}^2 = \left(\omega^2 - \frac{\mu_0 e^2 n_e c^2}{m}\right) > 0 \tag{232}$$

We then define the plasma frequency Ω as

$$\Omega^2 = \frac{\mu_0 e^2 n_e c^2}{m}. \tag{233}$$

Note that if n_e is too big then the wave is absorbed ($k^2 < 0$).

Emission or absorption of light by a free electron

Let's consider the 4-momentum of an electron $p^\mu = mu^\mu$ where the 4-velocity $u^\mu = \gamma(c, \mathbf{v})$ and $u^\mu u_\mu = -c^2$. On the other hand we can write $p^\alpha = (E/c, \mathbf{p})$ and hence

$$p^\alpha p_\alpha = -\frac{E^2}{c^2} + \mathbf{p}^2 = -m^2c^2. \tag{234}$$

In other words

$$E^2 = m^2c^4 + c^2\mathbf{p}^2 \tag{235}$$

and

$$E = \sqrt{m^2c^4 + c^2\mathbf{p}^2} \approx mc^2 + \frac{\mathbf{p}^2}{2m}. \tag{236}$$

Let's go back to our initial problem, and consider an absorption process:

$$p_\gamma + p_e = p_e \tag{237}$$

where a photon $p_\gamma = (E/c, 0)$ is absorbed by an electron. By squaring both sides of the former equation we obtain

$$(p_\gamma)^\mu(p_\gamma)_\mu + 2(p_\gamma)^\mu(p_e)_\mu + (p_e)^\mu(p_e)_\mu = (p'_e)^\mu(p'_e)_\mu \tag{238}$$

and since for the photon $(p_\gamma)^\mu(p_\gamma)_\mu = 0$ and for the electron $(p_e)^\mu(p_e)_\mu = (p'_e)^\mu(p'_e)_\mu = -mc^2$ where m is the rest mass, we obtain

$$(p_\gamma)^\mu(p_e)_\mu = 0 \tag{239}$$

and because there are no free indices left, this is a scalar and thus valid in all reference frames. In particular we can choose the reference frame in which the electron is initially at rest. This implies:

$$mE = 0 \tag{240}$$

This requires a photon of $E = 0$, i.e. this process is not possible.

L.19 Transformation of electromagnetic fields and relativistic Lagrangian

The first part of this lesson is dedicated to the transformations of electromagnetic fields [Jackson, Sect. 11.10]. As an example of transformation

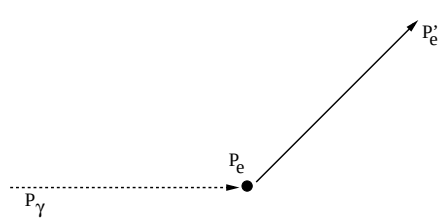


Figure L.8: *Impossibility of absorption of light by a free electron*

it is considered the one of an observer in a reference system K that sees a charge q moving by a straight-line path with a velocity $\mathbf{v}$*. The charge is at rest in the system K'. These transformations show that* $\mathbf{E}$ *and* $\mathbf{B}$ *fields have no independent existence. In the second part of the lecture, the Lagrangian of a charged particle in external fields is derived by writing down the action integral and by finding an invariant expression for the interaction part:* $-e/(\gamma c)u_\alpha A^\alpha$*. Hence the conjugate moment* p_α *is not only the kinetic one but contains also the vector potential contribution. Finally Lagrange (that corresponds to Lorentz's force) and Hamilton equations are derived [Jackson, Sect 12.1].*

How do $F^{\alpha\beta}$ transform under Lorentz's transformations ? The general expression (sometimes used as definition of a tensor) is:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta} \tag{241}$$

Under a Lorentz boost, the fields transform as below:

$$E'_x = E_x \qquad B'_x = B_x \tag{242}$$
$$E'_y = \gamma(E_y - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z) \tag{243}$$
$$E'_z = \gamma(E_z + \beta B_y) \qquad B'_z = \gamma(B_z - \beta E_y). \tag{244}$$

[to be updated]

Lagrangian of a charged particle

$$S = -\alpha\int_a^b ds = \int_{t_1}^{t_2} L dt \tag{245}$$

noe $ds = cd\tau$ and $d\tau = dt\sqrt{1-\frac{v^2}{c^2}}$, hence it follows:

$$L = -\alpha c\sqrt{1-\frac{v^2}{c^2}} \tag{246}$$

that represents the Lagrangian of a single charged particle. Now:

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \frac{\alpha\mathbf{v}/c}{\sqrt{1-\frac{v^2}{c^2}}} \tag{247}$$

and the energy is then

$$E = \mathbf{p}\cdot\mathbf{v} - L = \frac{\alpha c^2}{\sqrt{1-\frac{v^2}{c^2}}} \approx \alpha c^2 + \alpha\mathbf{v}^2/2 + ... \tag{248}$$

from here one can see that α is the rest mass of the particle. So we can write the Lagrangian as

$$L = -\frac{mc}{\gamma}\sqrt{-u_\alpha u^\alpha} \tag{249}$$

where the 4-velocity is $u^\alpha = (\gamma c, \gamma \mathbf{u})$. The actioni S reads:

$$\begin{aligned} S &= -mc\int_{\tau_1}^{\tau_2} d\tau\sqrt{-u_\alpha u^\alpha} && (250)\\ &= -mc\int_{\tau_1}^{\tau_2} d\tau\sqrt{\eta^{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}} && (251)\\ &= -mc\int_{s_1}^{s_2} \sqrt{\eta^{\alpha\beta}dx^\alpha dx^\beta} \end{aligned}$$

If an external field is present we have to add an interaction term to the Lagrangian:

$$L = -mc^2\sqrt{1-\frac{\mathbf{u}^2}{c^2}} - \frac{e}{\gamma c}u_\alpha A^\alpha \tag{252}$$

where $-\frac{e}{\gamma c}u_\alpha A^\alpha = -e\phi + \frac{e}{c}\mathbf{u}\cdot\mathbf{A}$ if then one computes the Lagrange equations it follows:

$$m\frac{d^2x^\alpha}{d\tau^2} = \frac{e}{c}(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\frac{dx^\beta}{d\tau}. \tag{253}$$

where $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. The former represents the expression of the Lorentz force. The conjugate moment is:

$$p^\alpha = \frac{\partial L}{\partial(\frac{dx_\alpha}{d\tau})} = \gamma m u^\alpha + \frac{e}{c}A^\alpha \tag{254}$$

Hamiltonian is

$$H = \frac{1}{2m}(p_\alpha - \frac{e}{c}A_\alpha)(p^\alpha - \frac{e}{c}A^\alpha) - \frac{1}{2}mc^2, \tag{255}$$

and hamilton equations are:

$$\begin{aligned} \frac{dx^\alpha}{d\tau} &= \frac{1}{m}(p^\alpha - \frac{e}{c}A^\alpha) && (256)\\ \frac{dp^\alpha}{d\tau} &= \frac{e}{mc}((p_\gamma - \frac{e}{c}A_\gamma)\partial^\alpha A^\gamma && (257) \end{aligned}$$

[to be updated]

L.20 Lagrangian for the electromagnetic field

In this last lecture of this course a general expression of the Lagrangian of the electromagnetic field is discussed for the case of specified external sources of charge and current [Jackson, sec. 12.7]. Equations of motion of the electromagnetic fields are then found by writing the Euler-Lagrange equations that correspond to the covariant form of the (inhomogeneous) Maxwell's equations. Next follows an explanation of the meaning of the time-time and time-space components of the canonical stress tensor [Jackson, eq. 12.103].The solution of the Wave equation in covariant form

$$\partial^\alpha \partial_\alpha A^\beta = \frac{4\pi}{c} J^\beta \tag{258}$$

is obtained through a k-space Green function approach that naturally leads to retarded and advanced solutions [Jackson, sec. 12.11]. As an application, Liénard-Wiechert potentials are then derived from the solution [Jackson, sec. 14.6]. Finally a covariant generalization of the Larmor's formula for 1 *is given and Liénard result is discussed by showing the angular distribution of radiation emitted by an accelerated charge (synchrotron radiation) [Jackson Sec 14.2 and 14.3].*

Let's write a general expression of the Lagrangian of the electromagnetic field:

$$L^{em} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha \tag{259}$$

Note that the former expression defines a scalar quantity and in particular

$$F_{\alpha\beta} F^{\alpha\beta} = cost(\mathbf{E}^2 - \mathbf{B}^2)$$

is a scalar. By writing explicitly $F^{\alpha\beta}$ in Eq. 260 we have:

$$L^{em} = -\frac{1}{16\pi} (\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) - \frac{1}{c} J_\alpha A^\alpha. \tag{260}$$

Now let's write the Lagrange equations for L^{em} given in Eq. 260. We first recall that for a generic Lagrangian

$$\int \mathcal{L}(\phi_k, \partial^\alpha \phi_k) d^3x$$

where $\phi_k(\mathbf{x})$ are fields, the Lagrange equations are:

$$\partial^\beta \frac{\partial \mathcal{L}}{\partial(\partial^\beta \phi_k)} = \frac{\partial \mathcal{L}}{\partial \phi_k} \tag{261}$$

Coming back to L^{em} we have

$$\partial^\beta \frac{\partial L^{em}}{\partial(\partial^\beta A_\alpha)} = \frac{\partial L^{em}}{\partial A_\alpha} \tag{262}$$

i.e.

$$\frac{1}{4\pi}\partial^\beta F_{\beta\alpha} = \frac{1}{c}J_\alpha \tag{263}$$

that are Maxwell equations (as a side remark, note that $A_\alpha = \eta_{\alpha\beta}A^\beta$ and for the zero component $A_0 = \eta_{00}A^0 = -A^0$.) Indeed

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta \tag{264}$$

may be written as

$$\Box A^\beta - \partial^\beta(\partial_\alpha A^\alpha) = \frac{4\pi}{c}J^\beta \tag{265}$$

where the D'Alambert operator is defined as

$$\Box = \partial^\alpha\partial_\alpha = -\frac{\partial^2}{\partial x^{02}} + \nabla^2. \tag{266}$$

In the Lorentz's gauge we finally obtain (wave equation)

$$\Box A^\beta = \frac{4\pi}{c}J^\beta \tag{267}$$

Energy-momentum tensor

$$\begin{aligned} T^{em}_{\alpha\beta} &= \frac{\partial L^{em}}{\partial(\partial_\alpha A^\lambda)}\partial^\beta A^\lambda - \eta^{\alpha\beta}L^{em} \\ &= \frac{1}{4\pi}[F^{\mu\alpha}F^\beta_\mu - \frac{1}{4}\eta^{\alpha\beta}F^{\mu\nu}F_{\mu\nu}] \end{aligned} \tag{268}$$

Let's analise the $\alpha = 0 = \beta$ component:

$$\int T^{00}\ d^3x = \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{B}^2)\ d^3x = E_{field} \tag{269}$$

i.e. the time-time component of the energy-momentum tensor gives the energy of the electromagnetic field. Let's now see the $\alpha = 0$ and $\beta = i$ (time-space) components:

$$\int T^{0i}\ d^3x = \frac{1}{4\pi}\int(\mathbf{E}\times\mathbf{B})\ d^3x = c\mathbf{P}_{field} \tag{270}$$

these gives the Poynting vector contribution, i.e c times the total momentum of the electromagnetic fields.

Solution of the wave equation in covariant form

Let's now find the solution of Eq. 267. First we look for the Green function $D(x, x')$ for the equation

$$\Box_x D(x, x') = \delta^4(x - x'). \tag{271}$$

Let's define $z^\alpha = x^\alpha - x'^\alpha$. In the absence of boundary surfaces we can rewrite the previous equation as:

$$\Box_z D(z) = \delta^4(z). \tag{272}$$

Now by Fourier transforming we obtain

$$-k \cdot k\tilde{D}(k) = 1$$

i.e.

$$\tilde{D}(k) = -\frac{1}{k \cdot k} \tag{273}$$

that leads to the Green function

$$D(z) = -\frac{1}{(2\pi)^4} \int d^4k \; \frac{e^{-ik\cdot z}}{k \cdot k}. \tag{274}$$

Note that $k \cdot z = k_0 z_0 - \mathbf{k} \cdot \mathbf{z}$ and $k_\alpha z^\alpha = \eta_{\alpha\beta} k^\alpha z^\beta$. Clearly we have singularities when $k_0^2 - \mathbf{k}^2 = 0$. We can split the integral by first integrating in k_0:

$$D(z) = -\frac{1}{(2\pi)^4} \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{z}} \int_{-\infty}^{\infty} dk_0 \; \frac{e^{-ik_0 z_0}}{k_0^2 - \mathbf{k}^2}. \tag{275}$$

For $z_0 > 0$ the integration (by res. theorem applied to the path shown in Fig. L.20) leads to

$$D(z) = \frac{\theta(z_0)}{(2\pi)^3} \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{z}} \frac{\sin kz_0}{k} \tag{276}$$

where $k = |\mathbf{k}|$. By solving the former integral we arrive to the *retarded solution*

$$D(z) = \frac{\theta(z_0)}{4\pi R} \delta(z_0 - R) \tag{277}$$

where $R = |\mathbf{z}|$. By reintroducing the original variables x and x' we have

$$D(x - x') = \frac{\theta(x_0 - x_0')}{4\pi R} \delta(x_0 - x_0' - R). \tag{278}$$

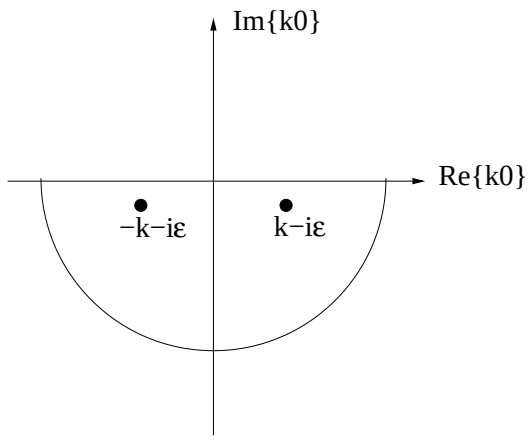


Figure L.9: *k_0-path used for calculating the integral in $D(z)$ described in the text for $z_0 > 0$.*

Similarly we can derive the *advanced Green function* ($z_0 < 0$):

$$D(x - x') = \frac{\theta(-x_0 + x_0')}{4\pi R}\delta(x_0 - x_0' + R). \tag{279}$$

The solution of the wave equation Eq. 267 can now be written in terms of the above defined Green functions (that can be written in covariant form see Jackson). For example:

$$A^\alpha(x) = A^\alpha_{in}(x) + \frac{4\pi}{c}\int d^4x' \; D(x - x')J^\alpha(x') \tag{280}$$

now $J^\alpha(x) = e\int d\tau \; u^\alpha(\tau)\delta^{(4)}(x - r(\tau))$ implies that

$$A^\alpha(x) = 2e\int d\tau \; u^\alpha(\tau)\theta(x_0 - r_0(\tau))\delta^{(4)}(x - r(\tau))^2 \tag{281}$$

to perform the integral we make use of

$$\delta(x - r(\tau))^2 = \frac{\delta(\tau - \tau_p)}{2[x - r(\tau)]_\beta u^\beta} \tag{282}$$

and we then obtain

$$A^\alpha(x) = \frac{e \; u^\alpha(\tau)}{u \cdot [x - r(\tau)]}|_{ret} \tag{283}$$

that represent the 4D form of the Liénard-Wichert potentials. By exploiting $u \cdot [x - r(\tau)] = u_0[x_0 - r_0(\tau)] - \mathbf{u} \cdot [\mathbf{x} - \mathbf{r}(\tau)]$ and $u^\alpha = \gamma(c, \mathbf{v})$ together with $R = |\mathbf{x} - \mathbf{r}(\tau)| = x_0 - r_0(\tau)$ we can write:

$$u \cdot [x - r(\tau_p)] = \gamma c R - \gamma \mathbf{v} \cdot \mathbf{R} = \gamma c R(1 - \beta \cdot \mathbf{n}) \tag{284}$$

and then it is easy to see that from Eq. 283 one can recover the non-covariant form of the Liénard-Wichert potentials already seen in the previous lessons, e.g

$$\phi = \frac{e}{R(1 - \beta \cdot \mathbf{n})}|_{ret} \tag{285}$$

Larmor formula

The Larmor formula for the radiated power states:

$$P = \frac{2}{3}\frac{e^2}{c^3}|\dot{\mathbf{v}}|^2 \tag{286}$$

Let's see a covariant generalization of this formula. As starting point we rewrite it as

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}. \tag{287}$$

Now

$$\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau} = (\frac{d\mathbf{p}}{d\tau})^2 - \frac{1}{c^2}(\frac{dE}{d\tau})^2$$

This leads to the Liénard result for the emitted power and for its angular distribution

$$P = \frac{2}{3}\frac{e^2}{c}\gamma^6[\dot{\beta}^2 - (\beta \times \dot{\beta})^2] \tag{288}$$

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c}\frac{[\mathbf{n} \times [(\mathbf{n} - \beta) \times \dot{\beta}]]^2}{(1 - \mathbf{n} \cdot \beta)^5} \tag{289}$$

Figure L.10: *Synchrotron radiation for* $\beta << 1$ *and* $\beta \sim 1$.